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Dynamic model of mesoscale eddies. Eddy parameterization for coarse resolution ocean circulation models

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In the framework of the eddy dynamic model developed in two previous papers (Dubovikov, M.S., Dynamical model of mesoscale eddies, *Geophys. Astophys. Fluid Dyn*., 2003, **97**, 311–358; Canuto, V.M. and Dubovikov, M.S., Modeling mesoscale eddies, *Ocean Modelling*, 2004, **8**, 1–30 referred as I–II), we compute the contribution of unresolved mesoscale eddies to the large-scale dynamic equations of the ocean. In isopycnal coordinates, in addition to the bolus velocity discussed in I–II, the mesoscale contribution to the large scale momentum equation is derived. Its form is quite different from the traditional down-gradient parameterization. The model solutions in isopycnal coordinates are transformed to level coordinates to parameterize the eddy contributions to the corresponding large scale density and momentum equations. In the former, the contributions due to the eddy induced velocity and to the residual density flux across mean isopycnals (so called Σ-term) are derived, both contributions being shown to be of the same order. As for the large scale momentum equation, as well as in isopycnal coordinates, the eddy contribution has a form which is quite different from the down-gradient expression.

Keywords: Density and momentum large scale equations; Isopycnal and level coordinates; Bolus and eddy-induced velocity; Residual density flux; Reynolds stress; Vorticity flux

1. Introduction

In two recent papers (Dubovikov, 2003; Canuto and Dubovikov, 2004) referred to below as I–II, a dynamic model for mesoscale eddies was presented whose main purpose was to parameterize the eddies in coarse resolution ocean general circulation models (OGCM) that cannot afford a resolution better than a few degrees. As Muller and Garrett (2004) have noticed, parameterizations of small-scale processes "must be specified as formulae rather than just numerical values". This general requirement is especially important for the parameterization of mesoscale eddies. The reason is

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that eddies are coherent structures and thus the eddy fluxes at a given depth are functions of large scale fields not only at the same depth, but at all other depths. It is clear that parameterizing such complex functions is quite difficult using the numerical solutions from eddy resolving models which are being developed extensively at present beginning with a pioneering study by Rix and Willebrand (1996). However, such models may be useful to validate model parameterizations. The latter may be tackled using either a phenomenological approach or some fundamental ‘‘microscopic’’ turbulence model. Thus far only the former approach was employed. The most impressive accomplishment is the well-known eddy induced velocity parameterization by Gent and McWilliams (1990, hereafter GM; see also Gent *et al*., 1995) which has considerably improved the performance of OGCMs in the description of the thermohaline circulation and temperature distribution (Danabasoglu *et al*., 1994; Bonning *et al*., 1995). On the other hand, the success of the GM model has given rise to new problems, such as the mesoscale diffusivity dependence on the large scale fields, the fulfillment of boundary conditions and the completeness of the model. The situation with eddy parameterizations in large scales momentum equation is worse. In fact, on the one hand, it is believed that the eddy contribution to the momentum equation is due to the baroclinic instability, but on the other hand, the default sub-grid mesoscale parameterizations in most OGCMs is still a diffusion of momentum with the eddy viscosity determined by computational stability rather than by physical arguments. As McWilliams (1996) pointed out, ‘‘*Evidently the default sub-grid-scale mesoscale parameterization form of horizontal momentum diffusion is inadequate, for it fails to represent the qualitatively important feature of wind-driven gyres*.’’ As alternatives to the default parameterization, it is worth referring to the phenomenological models by Gent and McWilliams (1996) based on the Eliassen–Palm fluxes and by Greatbach (1998) based on the diffusion of potential vorticity. The model developed in I–II is an alternative to the phenomenological approach. It is based on the dynamic turbulence model developed by the authors in a series of papers listed in I. The model is derived from general physical principles, does not contain adjusting parameters and is not geared toward any specific flow. It was validated by testing against about 100 turbulent statistics in a wide variety of flows.

To study the isopycnal ocean turbulence, in I–II, the general CD model was applied to shallow water flows with small Froude and Rossby numbers. The major outcome of the application is the generation of mesoscale eddies which, according to Richardson (1993), ‘‘... are water-mass anomalies that have nearly circular flow around their centers and that survive for many rotations and may move through the background water at speeds and directions inconsistent with background flow’’. The equations for the eddy fields reduce to a vertical eigen-value problem, the real part of the eigen-value yields the eddy size, while the imaginary part yields the eddy drift velocity. The former was estimated to be of the order three times the deformation radius in agreement with observations (Stammer, 1997, 1998). The equations for eddy fields allow us not only to compute different eddy fluxes but also to follow the energy transfer from sources to sinks. By analyzing the latter problem we derived not only an ocean analog of the Lorenz diagram of energy exchange between eddies and large scales established in the atmosphere from observations (Holton, 1992, pp. 341, 342), but we also have worked out the details of the transfer within eddies. From the practical viewpoint, model dynamic equations allow us to parameterize the eddy fluxes to be used in coarse resolution OGCM’s. In I–II only the bolus velocity in isopycnal coordinates was computed since it is crucial in the analysis of fundamental problems such as the energy exchange between large scales and eddies.

The purpose of the present paper is two-fold. First, we present the details of the model technique for computing eddy fluxes. Second, we present the complete set of expressions for eddy fluxes in terms of large scale fields which is required for mesoscale eddy parameterizations in coarse resolution OGCMs. We complete the solution of the problem in isopycnal coordinates in the mean momentum equations (section 2.2). In addition, we find more accurate approximations for eddy kinetic energy and eddy production than in I–II (section 2.3). The latter result is almost model independent (only the horizontal locality of the production is assumed that is justified within the present model). Besides, we develop a somewhat different approach to analyze the dissipation of eddy energy which allows us to obtain eddy parameterization containing only one global (flow independent) parameter $C_d \sim 1$. In section 3 we consider eddy parameterization in level coordinates. The transformation from isopycnal to level coordinates is a non-trivial problem due to the random nature of isopycnal surfaces (McDougall,1998; McDougall and McIntosh, 2001). In particular, starting from GM, the density equation accounts for the eddy induced velocity but neglects the residual density flux across mean isopycnals Σ (see, e.g., Treguier *et al.*, 1997). However, as shown by the simulations of Gille and Davis (1999), the contribution of Σ is important. The present model confirms this conclusion (see section 3.1.2) and yields a parameterization of Σ in terms of large scale fields. In this relation, it is worth recalling that in the framework of the temporal-residual-mean velocity formalism (McDougall and McIntosh, 1996, 2001), the Σ-term does not appear. In fact, in this approach, vertical coordinates are the large scale mean heights of isopycnal surfaces. For this reason, the corresponding dynamics is equivalent to that in isopycnal coordinates. As for eddy parameterization in the mean momentum equation, the results in both isopycnal and level coordinates are quite different from the default down-gradient parameterization confirming the above remark by McWilliams (1996) (see Sections 2.2 and 3.2).

2. Eddy parameterization in isopycnal coordinates

2.1 *Review of model dynamic equations*

In I–II we derived the eddy dynamic equations in isopycnal coordinates in Fourier representation within horizontal (isopicnal) surfaces (i.e., considering eddy fields as functions of $2D$ wave-vector $\mathbf{q}$ and of the vertical coordinate which is the potential density ρ). It was shown that in the vicinity of the maximum of the energy spectrum at wave-numbers $q \sim q_0$, the equations yield coherent structures which are the mesoscale eddies. In that region, all eddy fields may be expressed in term of one of them, say, the mesoscale Bernoulli potential B'. Specifically, the eddy contributions to the layer thickness, h', and to the height of an isopycnal surface, z', are expressed through B' in equations (I.14d) and (II.4c). As for the eddy velocity field $\mathbf{U}'(\mathbf{q},\rho)$, it was decomposed into solenoidal (divergence free) and potential (curl free) components s and $\tilde{s}$ (sec (I.5e,f) and (II.4g)), which are expressed in terms of B' in equations (I.14c,a) and (II.10a,b). Equations (I.5f,14c) and (II.4g,10a,b) imply that the field $\mathbf{s}(\tilde{\mathbf{s}})$ is the geostrophic (ageostrophic) component of the eddy velocity in Fourier space. The field B' was found by solving the eigen-value problem (equations (I.15,16) and (II.11)), with boundary conditions (I.19a,d). The solution of the eigen value problem is simplified due to the following hierarchy of time scales:

$$q_0\bar{u} < q_0 K^{1/2} \sim \tilde{\nu} \sim \tilde{\chi} < f \tag{1a}$$

which allowed us to solve equations (I.15,16) perturbatively expanding the solution in powers of $\Omega_1\tau, \Omega_2\tau$. In fact, if we adopt a Gaussian approximation for eddy spectra in the vicinity of their maxima q_0[1], from equation (I.8b) we obtain

$$\tilde{\nu} = \frac{1}{2} q_0 K^{1/2}, \tag{1b}$$

where K is eddy kinetic energy averaged within an isopycnal surface.[2] From (1b) and (I.16c–f,20,25m) (or (II.11d,e)) we deduce that

$$\Omega_1\tau \sim \Omega_2\tau \sim \bar{u}/K^{1/2} < 1, \tag{1c}$$

which justifies the application of a perturbative approach to solve the eigen-value problem (I.15,16). Even though the parameters (1c) are not so small, the perturbative approach is applicable since corrections to the zeroth order approximation turn out to be of the second order in $\Omega_{1,2}\tau$ (this is because variables (1c) are multiplied by i in (I.16a), (II.11b)). By solving the eigen-value problem in the zeroth order, q_0 and the eddy size were expressed in terms of the Rossby deformation radius, (see (I.25g), (II.13b)). In the first order, we found the eddy frequency and eddy drift velocity, (see (I.20, 25m,n), (II.15c–e)). Below we quote these results since we shall often refer to them[3]:

$$q_0^{-1} = \sigma_t^{1/2} r_d, \qquad \omega_1 = \mathbf{q} \cdot \mathbf{u}_d, \tag{1d}$$

$$\mathbf{u}_d = \langle \overline{\mathbf{u}} \rangle + (1+\sigma_t)^{-1}\mathbf{c} - f r_d^2 (1+\sigma_t^{-1})^{-1} \mathbf{e}_\rho \times \langle \bar{h}^{-1} \nabla_\rho \bar{h} \rangle, \tag{1e}$$

where $\mathbf{c}$ is the velocity of the barotropic Rossby waves and the averaging operation $\langle A \rangle$ is defined as

$$\langle A \rangle \equiv \left[\int_{\rho_s}^{\rho_b} \Gamma^{1/2}(\rho) \overline{h}\, \mathrm{d}\rho \right]^{-1} \int_{\rho_s}^{\rho_b} A(\rho)\Gamma^{1/2}(\rho)\bar{h}\, \mathrm{d}\rho, \tag{1f}$$

where

$$\Gamma(\rho) = K(\rho)/K_s \tag{1g}$$

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[1]In I, II a Kolmogorov form for the energy spectrum was adopted;this could create the impression that the results critically depend on this assumption whereas in reality for different spectra, the results change not more than 10–20%.
[2]In I, II we used the notation K for the kinetic energy within the eddy whereas the averaged one equals $F_e K$ where F_e is the eddy filling factor. In the present paper we give up F_e since it results in apparent additional free parameter in eddy parameterization in coarse resolution computations.
[3]In I, II we omitted the subscript ρ in ∇_ρ.

is the normalized profile of eddy kinetic energy (K_s is the surface kinetic energy). The notation ω_1 in (1d) implies that the result is of the first order in $\Omega_{1,2}\tau$. Since in the vicinity of q_0, eddy fields can be expressed in terms of one of them, in this region the spectra of the eddy second-order moments (e.g. eddy fluxes) can be expressed in terms of one of them, say, the energy spectrum $E(q_0)$. Assuming that the spectra are concentrated in the vicinity of q_0 and that their widths are more or less similar, one can express the eddy fluxes in terms of eddy kinetic energy K. In I, II we computed the thickness flux related to the bolus velocity (equations (I.28) and (II.17a,b)), which plays a central role in the energy exchange between mesoscale eddies and large scale fields. Below we quote the final results with the correction commented in footnotes[1,2]:

$$\mathbf{u}_* \equiv (\bar{h})^{-1}\overline{h'\mathbf{u}'} = -\kappa_h \mathbf{S}, \tag{1h}$$

$$\mathbf{S} = \bar{h}^{-1}\nabla_\rho \bar{h} - \left\langle \bar{h}^{-1}\nabla_\rho \bar{h}\right\rangle + r_d^{-2} f^{-1}(1+\sigma_t^{-1})\mathbf{e}_\rho \times (\mathbf{u} - \langle \mathbf{u}\rangle), \tag{1i}$$

$$\kappa_h(\rho) = 2\sigma_t^{3/2} r_d K^{1/2}(\rho). \tag{1j}$$

2.2 *Eddy parameterization in the mean momentum equation*

In the large scale momentum equations, the mesoscale contributions are represented by the adiabatic and diabatic momentum fluxes $\mathbf{A}^a$ and $\mathbf{A}^d$ formed by isopycnal and diapycnal turbulence which are well separated in wave-numbers within isopycnal surfaces. The former occurs at eddy scales $\sim r_d$ whereas the latter occurs at $l_1 \sim 10^2 - 10^3$ m. The present eddy model considers only the adiabatic term $\mathbf{A}^a$ which is usually parameterized as a down-gradient term (Bleck, 1998, 2002). Thus, we adopt the adiabatic approximation for the momentum equation, (see equation (I.2a)), which we rewrite as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla_\rho\left(|\mathbf{u}|^2\right) + (f+\zeta)\mathbf{e}_\rho \times \mathbf{u} = -\frac{1}{\rho}\nabla_\rho B, \tag{2a}$$

where ζ is the kinematic vorticity

$$\zeta = \left(\nabla_\rho \times \mathbf{u}\right)\cdot \mathbf{e}_\rho. \tag{2b}$$

Splitting the velocity field into large scale and turbulent components, we obtain the equation for the former

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \left(f + \bar{\zeta}\right)\mathbf{e}_\rho \times \mathbf{u} = -\nabla_\rho \bar{B} - \frac{1}{2}\nabla_\rho |\bar{\mathbf{u}}|^2 - \mathbf{A}^a, \tag{2c}$$

where

$$\mathbf{A}^a = \nabla_\rho K + \mathbf{e}_\rho \times \mathbf{F}_{\zeta'} \tag{2d}$$

and $\mathbf{F}_{\zeta'}$ is the flux of the fluctuating kinematic vorticity

$$\mathbf{F}_{\zeta'} = \overline{\zeta' \mathbf{u}'} \tag{3a}$$

which, much as K, must be expressed in terms of large scale fields.

To compute the flux $\mathbf{F}_{\zeta'}$, we split it into the geostrophic and ageostrophic components

$$\mathbf{F}_{\zeta'} = \mathbf{F}_g + \mathbf{F}_{ag}, \qquad \mathbf{F}_g = \overline{\zeta' \mathbf{u}'_g}, \qquad \mathbf{F}_{ag} = \overline{\zeta' \mathbf{u}'_a}. \tag{3b}$$

Then $\mathbf{A}^a$ (2d) is split

$$\mathbf{A}^a = \mathbf{A}_g + \mathbf{A}_{ag}, \qquad \mathbf{A}_g = \nabla_\rho K + \mathbf{e}_\rho \times \mathbf{F}_g, \qquad \mathbf{A}_{ag} = \mathbf{e}_\rho \times \mathbf{F}_{ag}. \tag{3c}$$

Since $\mathbf{u}'_g$ is divergence free and ζ' (2b) is contributed only by $\mathbf{u}'_g$, one can derive from (3b)

$$\mathbf{e}_\rho \times \mathbf{F}_g = \nabla_\rho \cdot [\mathbf{R}_g], \tag{3d}$$

where $[\mathbf{R}_g]$ is the traceless component of the geostrophic Reynolds stress

$$\mathbf{R}_g = \overline{\mathbf{u}'_g \mathbf{u}'_g}. \tag{3e}$$

In (3d) we adopt the following notation for an arbitrary tensor $\mathbf{A}$

$$[\mathbf{A}] \equiv \mathbf{A} - \frac{1}{2} \boldsymbol{\delta}_\rho \mathrm{Trace}\, \mathbf{A}, \tag{3f}$$

where $\boldsymbol{\delta}_\rho$ is the $2D$ Kroneker tensor in an isopycnal surface. The expression for $\mathbf{A}_g$ in (3c,d) together with

$$\mathbf{R}_g = \boldsymbol{\delta}_\rho K + [\mathbf{R}_g] \tag{3g}$$

may be presented in the more compact form

$$\mathbf{A}_g = \nabla_\rho \cdot \mathbf{R}_g. \tag{3h}$$

It is worth noticing that in the framework of the present model, it is difficult to compute $\mathbf{F}_g$ directly from the definition (3b) since the model yields all results as expansions in powers of the small parameter $\alpha = r_d/L$, where L is the length scale of mean fields. At the present stage, we can compute only terms of the zero power of α. But such a contribution to $\mathbf{F}_g$ vanishes, as it follows from equation (3d), in the limit $L \to \infty$ that implies $\nabla_\rho \to \mathbf{0}$ when applied to mean fields and mean momenta like $\mathbf{R}_g$. We confirm this conclusion below from a different analysis. Thus, instead of computing $\mathbf{e}_\rho \times \mathbf{F}_g$, we compute $\mathbf{R}_g$. Details of the computation are presented in Appendix A. Below we present the result for $[\mathbf{R}_g]$ which is contributed mostly by the barotropic component which is of the second order in the small parameter (1c):

$$[\mathbf{R}_g] = \frac{1}{2}(1+2c_1)[\mathbf{I}_1\mathbf{I}_1] + \frac{1}{2}[\mathbf{I}_1\mathbf{I}_2 + \mathbf{I}_2\mathbf{I}_1] - [\mathbf{T}_1] - [\mathbf{T}_2]. \tag{4a}$$

where the vectors $\mathbf{I}_{1,2}$ and tensors $\mathbf{T}_{1,2}$ are given by

$$\mathbf{I}_1 = c_2\langle \mathbf{S}\,\mathrm{sgn}B_1\rangle_0, \qquad \mathbf{I}_2 = -c_2\left\langle \Gamma^{-1/2}\,\mathbf{S}\right\rangle_0, \tag{4b}$$

$$\mathbf{T}_1 = \left\langle(\bar{\mathbf{u}} - \mathbf{u}_d)(\bar{\mathbf{u}} - \mathbf{u}_d)\Gamma^{-1/2}\,\mathrm{sgn}B_1\right\rangle_0, \tag{4c}$$

$$\mathbf{T}_2 = -c_1\left\langle(\bar{\mathbf{u}} - \mathbf{u}_d)(\bar{\mathbf{u}} - \mathbf{u}_d)\right\rangle_0, \tag{4d}$$

$$c_1 = \left\langle \Gamma^{1/2}\right\rangle_0^{-1}\langle \mathrm{sgn}B_1\rangle_0, \qquad c_2 = 2\sigma_t f r_d^2, \tag{4e}$$

where $\mathbf{S}$ and $\mathbf{u}_d$ are given in (1i,e), and the averaging operation $\langle X\rangle_0$ is defined for any function $X(\rho)$ as

$$\langle X\rangle_0 \equiv -H^{-1}\int_{\rho_s}^{\rho_b} X(\rho)\bar{h}\,\mathrm{d}\rho, \tag{4f}$$

where H is the ocean depth.

Next, we compute $\mathbf{A}_{ag}$ employing equations (3c,b), (2b). To this end, we begin by deriving the expression for the ageostrophic vorticity flux $\tilde{\mathbf{F}}_{ag}(\mathbf{q})$ in $\mathbf{q}$-space taking into account that here the ageostrophic velocity coincides with the component $\tilde{\mathbf{s}}(\mathbf{q})$:

$$\delta(\mathbf{q}-\mathbf{q}')\tilde{\mathbf{F}}_{ag}(\mathbf{q}) = \mathrm{Re}\left\{\overline{\zeta'(\mathbf{q})\tilde{\mathbf{s}}^*(\mathbf{q}')}\right\}. \tag{5a}$$

From the definition of the kinematic vorticity (2b) in $\mathbf{q}$-space we get

$$\zeta'(\mathbf{q}) = \mathrm{i}\mathbf{q}\times\mathbf{u}'(\mathbf{q})\cdot\mathbf{e}_\rho. \tag{5b}$$

Substituting the decomposition (I.5e,f), we obtain

$$\zeta'(\mathbf{q}) = -\mathrm{i}qs(\mathbf{q}) \tag{5c}$$

i.e., only the geostrophic component of the eddy velocity field contributes to ζ'. From equation (5c) it follows that because of the imaginary factor i, the result for $\tilde{\mathbf{F}}_g(\mathbf{q})$ obtained by substituting $\tilde{\mathbf{s}}\to\mathbf{s}$ in (5a), equals zero. However, as we have discussed after (3h), this conclusion is valid only in the limit $L\to\infty$. That is why we have computed $\mathbf{F}_g$ in the different way.

Next, we substitute (5c) into (5a). With (I.14a) we obtain

$$\delta(\mathbf{q}-\mathbf{q}')\tilde{\mathbf{F}}_{ag}(\mathbf{q}) = -f^{-1}q_0\mathbf{n}\mathbf{q}\cdot(\bar{\mathbf{u}} - \mathbf{u}_d + \mathbf{c})\,\overline{s(\mathbf{q})s^*(\mathbf{q}')}. \tag{5d}$$

Let us integrate this equation over $\mathbf{q}'$ and take into account the definition of the kinetic energy density in $\mathbf{q}$-space

$$\delta(\mathbf{q}-\mathbf{q}')\tilde{E}(\mathbf{q}) = \frac{1}{2}\overline{u'(\mathbf{q})u'^{*}(\mathbf{q}')} \approx \frac{1}{2}\overline{s(\mathbf{q})s^{*}(\mathbf{q}')}. \tag{5e}$$

We obtain

$$\tilde{\mathbf{F}}_{ag}(\mathbf{q}) = -2f^{-1}q_0\mathbf{n}\mathbf{q}\cdot(\bar{\mathbf{u}}-\mathbf{u}_d+\mathbf{c})\tilde{E}(\mathbf{q}). \tag{5f}$$

Finally, integrating (5f) over $\mathbf{n}$ and using the general definition of spectra

$$F(q) = q\int \tilde{F}(\mathbf{q})\,\mathrm{d}\mathbf{n}, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|, \tag{5g}$$

we derive

$$\mathbf{F}_{ag}(q) = -q_0^2 f^{-1}(\bar{\mathbf{u}}-\mathbf{u}_d+\mathbf{c})E(\mathbf{q}). \tag{6a}$$

In arriving at this result, we have taken into account the approximate axi-symmetry of eddy fields, specifically $\tilde{E}(\mathbf{q}) = \tilde{E}(q)$, and that, for any constant vector $\mathbf{A}$,

$$\int \mathbf{n}\mathbf{n}\cdot\mathbf{A}\,\mathrm{d}\mathbf{n} = \pi\mathbf{A}. \tag{6b}$$

Recall that (5d,f) and therefore (6a) are valid only in the vicinity of the maxima of the spectra, i.e., at $q \approx q_0$. Assuming that the main contributions to $\mathbf{F}_{ag}$ and to the eddy kinetic energy K come from the maxima of the spectra and taking into account (1d) and (I.6e), we obtain the final result

$$\mathbf{F}_{ag} = \left[\beta f^{-1}\mathbf{e}_x - \left(\sigma_t r_d^2 f\right)^{-1}(\bar{\mathbf{u}}-\mathbf{u}_d)\right]K. \tag{6c}$$

Thus, the final result for $\mathbf{A}^a$ following from (3c,h), (6c) is

$$\mathbf{A}^a = \nabla_\rho\cdot\mathbf{R}_g + \left(\beta f^{-1}\mathbf{e}_y + \left(\sigma_t r_d^2 f\right)^{-1}(\bar{\mathbf{u}}-\mathbf{u}_d)\times\mathbf{e}_\rho\right)K, \tag{6d}$$

which expresses the isopycnal eddy momentum flux in terms of large scale fields (with account of (4)) and the eddy kinetic energy. The dominating term in (6d) is $\nabla_\rho K$. We evaluate its contribution by substituting $|\nabla_\rho| \sim 1/L$ and typical values $L \sim 10^6$m and $K \sim (10^{-2}-10^{-3})\,\mathrm{m}^2\mathrm{s}^{-2}$ to obtain $(10^{-8}-10^{-9})\,\mathrm{ms}^{-2}$. The very presence of this term does not depend on mesoscale modelling. Meanwhile the default parameterizations of $\mathbf{A}^a$ is the down-gradient model (Bleck, 1998, 2002) in which the sub-grid turbulent viscosity is chosen on the basis of computational stability (Gent and McWilliams, 1996). Bleck's (2002) choice of the turbulent viscosity $\sim 10^4\mathrm{m}^2\mathrm{s}^{-1}$

yields $|\mathbf{A}^a| \sim 10^{-10}\,\mathrm{ms}^{-2}$ which is much smaller than (6d) and has quite a different functional structure.

To apply the results (1h–j), (4a–e), (6d) to course resolution OGCMs, one still needs to express the eddy kinetic energy K, its surface value K_s and the normalized profile Γ in terms of large scale fields. Because of the importance of this problem, we solve it in the next sections more accurately than in papers I, II.

2.3 *Eddy Kinetic Energy in Terms of Large Scale Fields*

As discussed in I, II, the value of K is determined from the equilibrium between the processes of eddy production by large scale potential energy at scales $\sim r_d$ and eddy energy dissipation at scales $l_1 < 1\,\mathrm{km}$. The analysis of these processes requires quite a different approach which we develop in the subsequent sections.

2.3.1 Production of Eddy Energy. We begin the analysis by considering the evolution of the eddy potential energy

$$W = \frac{1}{2}N^2\overline{z'^2}, \tag{7a}$$

where N is the Brunt-Waisala frequency. The analysis requires the evolution equation for $z(\rho)$ which in the adiabatic approximation is

$$\frac{\partial z}{\partial t} = w - \mathbf{u}\cdot\nabla_\rho z, \tag{7b}$$

where w is the vertical component of the velocity. Splitting all fields into mean and fluctuating (eddy) components, we deduce

$$\frac{\partial z'}{\partial t} = w' - \bar{\mathbf{u}}\cdot\nabla_\rho z' - \mathbf{u}'\cdot\nabla_\rho \bar{z} - \left(\mathbf{u}'\cdot\nabla_\rho z' - \overline{\mathbf{u}'\cdot\nabla_\rho z'}\right). \tag{7c}$$

Multiplying this equation by z' and averaging, we obtain

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{z'^2}\right) + \bar{\mathbf{u}}\cdot\nabla_\rho\left(\frac{1}{2}\overline{z'^2}\right) + \frac{1}{2}\overline{\mathbf{u}'\cdot\nabla_\rho z'^2} = \overline{w'z'} - \overline{\mathbf{u}'z'}\cdot\nabla_\rho\bar{z}, \tag{7d}$$

With accuracy to the factor ρ/g, the variable $(\overline{z'^2}/2)$ is the eddy potential energy per unit interval of density and per unit horizontal area. Thus, the second and third terms in the left hand side of (7d) represent the advection and diffusion of eddy potential energy whereas the right hand side represents production. As McDougall and McIntosh (2001) have argued, the diffusion term is much smaller than the advection since the former is a third order term in fluctuating fields. In Appendix C we show that the advection term is much smaller than the production. This implies that the production of eddy potential energy is local, i.e., its rate is determined by the production in the same place. Neglecting the second and third terms in the left hand side of equation (7d), multiplying the equation by N^2 and using (7a),we get

$$\frac{\partial W}{\partial t} = N^2\left(\overline{w'z'} - \overline{\mathbf{u}'z'}\cdot\nabla_\rho\bar{z}\right), \tag{7e}$$

where we have assumed that the characteristic time of variation of N is longer than that of the eddy potential energy.

Before computing the production of eddy kinetic energy, from equation (7a) we deduce the expression for w' in terms of the isopicnal component $\mathbf{u}'$ which we will refer to henceforth. Differentiating (7b) with respect to ρ and subtracting the evolution equation for $h = z_\rho$ (I.2b) we obtain the relation

$$w_\rho = -h\nabla_\rho\cdot\mathbf{u}, \tag{8a}$$

which is the isopycnal counterpart of the continuity relation $w_z = -\nabla_H\cdot\mathbf{u}$ in z-coordinates. To write the corresponding relation for the eddy components, we employ the condition which we also will often refer to henceforth

$$\frac{h'}{\bar{h}} = z'_z \sim \frac{z'(\rho)}{\widetilde{H}} \ll 1. \tag{8b}$$

In fact, z' estimates the characteristic variation of a level $z(\rho)$ within eddies which is approximately 10^2 m (Richardson, 1993), whereas $\widetilde{H} > 10^3$ m. Therefore, the ratio in (8b) is ~ 0.1. In practice, it becomes even smaller when we consider the filling factor of the eddies within a given flow. The latter is about few tens of a percent. Then

$$\overline{z'^2} \sim 10^3\mathrm{m}^2 \tag{8c}$$

and the ratio in (8b) is $\sim 3\cdot 10^{-2}$. Using condition (8b) and splitting (8a) into mean and fluctuating components, we obtain

$$w'_\rho = -\bar{h}\nabla_\rho\cdot\mathbf{u}'. \tag{8d}$$

To analyze the eddy kinetic energy production, we multiply equation (I.4a) for the eddy velocity by $\mathbf{u}'$ and average. We obtain

$$\partial_t\left(\frac{1}{2}\overline{|\mathbf{u}'|^2}\right) + \bar{\mathbf{u}}\cdot\nabla_\rho\left(\frac{1}{2}\overline{|\mathbf{u}'|^2}\right) + \frac{1}{2}\overline{\mathbf{u}'\cdot\nabla_\rho|\mathbf{u}'|^2} = -\rho^{-1}\overline{\mathbf{u}'\cdot\nabla_\rho B'}. \tag{9a}$$

By analogy with equation (7d) for eddy potential energy, the second and third terms on the left hand side of (9a) yield the advection and diffusion of eddy kinetic energy which are small in comparison with the right hand side which yields the production of K, i.e. the production of K is also local. This is shown in Appendix C. Next, we transform the production term (which is contributed by the ageostrophic component of eddy velocity $\mathbf{u}'_a$ only) as follows:

$$\overline{\mathbf{u}'_a \cdot \nabla_\rho B'} = \nabla_\rho \cdot \overline{\mathbf{u}'_a B'} + \bar{h}^{-1} \overline{\frac{\partial w'}{\partial \rho} B'} = \nabla_\rho \cdot \overline{\mathbf{u}'_a B'} + \bar{h}^{-1} \frac{\partial}{\partial \rho} (\overline{w' B'}) - g\bar{h}^{-1} \overline{w' z'}, \tag{9b}$$

where we used equation (8d) and the first relation in (I.6d).

Substituting equation (9b) into (9a), we notice that the first term in the right hand side of (9b) may be neglected since it is of the same order that the second term in the left hand side of (9a) because, as we show in Appendix C,

$$\mathbf{F}_B \equiv \overline{\mathbf{u}'_a B'} = \rho(\bar{\mathbf{u}} - \mathbf{u}_d + \mathbf{c})K. \tag{9c}$$

Thus, in the local approximation from equations (9a–c), we obtain

$$\frac{\partial K}{\partial t} = -(\rho \bar{h})^{-1} \frac{\partial}{\partial \rho} \left(\overline{w' B'} \right) - N^2 \overline{w' z'}. \tag{9d}$$

Summing equations (7e) and (9d), we get

$$\frac{\partial E_{\text{tot}}}{\partial t} = -(\rho \bar{h})^{-1} \frac{\partial}{\partial_\rho} \left(\overline{w' B'} \right) - N^2 \overline{\mathbf{u}' z'} \cdot \nabla_\rho \bar{z}, \tag{10a}$$

where $E_{\text{tot}} \equiv K + W$ is the total eddy energy. Multiplying this equation by $-\bar{h} \mathrm{d}\rho$ and integrating over the whole depth, we obtain $\mathcal{P}_e$, the column production of the total eddy energy per the unit horizontal area. The integration of the first term in r.h.s. yields zero since $w' = 0$ at the surface and bottom. Thus, we have

$$\mathcal{P}_e = -\rho^{-1} \int_{\rho_s}^{\rho_b} \overline{\mathbf{u}' z'} \cdot \nabla_\rho \bar{B}_\rho \, \mathrm{d}\rho = \rho^{-1} \int_{\rho_s}^{\rho_b} \nabla_\rho \bar{B} \cdot \frac{\partial}{\partial \rho} \left(\overline{\mathbf{u}' z'} \right) \mathrm{d}\rho, \tag{10b}$$

where we have used again the first relation (I.6d) and accounted for the boundary condition $z' = 0$, (I.19a,d), at the surface and bottom. Substituting the geostrophic relation between the mean velocity and $\nabla_\rho \bar{B}$, we obtain the final result

$$\mathcal{P}_e = -f \int_{\rho_s}^{\rho_b} \mathbf{e}_\rho \times \bar{\mathbf{u}} \cdot \mathbf{u}_p \bar{h} \, \mathrm{d}\rho, \tag{11a}$$

where we introduce

$$\mathbf{u}_p = \bar{h}^{-1} \frac{\partial}{\partial \rho} \left(\overline{\mathbf{u}' z'} \right), \tag{11b}$$

which can be called the eddy production velocity and which relates to the bolus velocity (1h) as follows:

$$\mathbf{u}_p = \mathbf{u}_* + \mathbf{u}_{**}, \tag{12a}$$

where

$$\mathbf{u}_{**} = h^{-1}\overline{z'\,\mathbf{u}'_\rho}. \tag{12b}$$

Using the geostrophic relation

$$\mathbf{u}'_\rho = (g/f\rho_0)\left[\mathbf{e}_\rho \times \nabla_\rho z'\right], \tag{13a}$$

(12b) becomes

$$\mathbf{u}_{**} = -(N^2/f)\left[\mathbf{e}_\rho \times \nabla_\rho\left(\frac{1}{2}\overline{z'^2}\right)\right]. \tag{13b}$$

When estimating (13b) with $f \sim 10^{-4}\mathrm{s}^{-1}$, the typical values for $|\nabla_\rho| \sim L^{-1} \sim 10^{-6}\,\mathrm{m}^{-1}$, $N^2 \sim 10^{-5}\mathrm{s}^{-2}$ and (8c), we obtain $|\mathbf{u}_{**}| \sim 10^{-4}\,\mathrm{m\,s}^{-1}$ which is smaller than $\mathbf{u}_* \sim 3 \cdot 10^{-3}\,\mathrm{m\,s}^{-1}$ at least in order. The former evaluation looks unexpected since when differentiating in (11b), both resulting terms are seemingly of the same order. The reason for the reduction of the second term in (12a) is the weak correlation in (12b). This fact comes about quite directly. In fact, while in (13a) $|\nabla_\rho| \sim r_d^{-1} \sim 3 \cdot 10^{-5}\,\mathrm{m}^{-1}$, in (13b) in the limit of the homogeneous flow $|\nabla_\rho| \to 0$ which in real flows is replaced by $|\nabla_\rho| \sim L^{-1} \sim 10^{-6}\,\mathrm{m}^{-1}$. The above evaluation of $\mathbf{u}_{**}$ is confirmed by computing (12b) in the framework of the present model with use of (13b) and (7a)

$$\mathbf{u}_{**} = -(N^2/f)\left[\mathbf{e}_\rho \times \nabla_\rho\left(W/N^2\right)\right], \tag{13c}$$

which yields the same $|\mathbf{u}_{**}| \sim 10^{-4}\,\mathrm{m\,s}^{-1}$ with accounting for that $W \sim K$. Thus, in (12a) the second term can be omitted adopting instead

$$\mathbf{u}_p \approx \mathbf{u}_*. \tag{13d}$$

Still, in Appendix D we obtain the expression of W (and therefore $\mathbf{u}_{**}$) in terms of large scale fields and the surface value K_s (which, in turn, is expressed in terms of large scale fields in the end of the present section):

$$W = \sigma_t\left(\rho_0 r_d f g^{-1}\right)^2 N^2\left[\frac{\partial}{\partial\rho}(B_1(\rho))\right]^2 K_s, \tag{13e}$$

where $B_1(\rho)$ is the eigen-function of (I.25b) normalized $B_1(\rho_s) = 1$.

Let us evaluate a typical column eddy production by substituting approximation (13d) into (11a) together with the typical values $\bar{u} \sim 10^{-2}\,\mathrm{ms}^{-1}$, $u_* \sim 10^{-3}\,\mathrm{ms}^{-1}$ and the dynamical ocean depth $\sim 1\,\mathrm{km}$. Then we obtain $\mathcal{P}_e \sim 10^{-6}\,\mathrm{m}^3\mathrm{s}^{-3}$ that corresponds to the global eddy production $\sim 0.3\mathrm{TW}$ to be compared with the rate of a global work done by the wind $\sim 1\mathrm{TW}$ (Wunsch, 1998).

To interprete relation (11a), it is convenient to rewrite it in the approximation (13d). Substituting (1h–j), after simple manipulations and use of (1f), we obtain

$$\mathcal{P}_e = K_s^{1/2} f r_d (\phi_1 + \phi_2), \tag{14a}$$

$$\phi_1 = 2\sigma_t^{3/2} \int_{\rho_s}^{\rho_b} \Gamma^{1/2}(\rho) \mathbf{e}_\rho \times (\bar{\mathbf{u}} - \langle \bar{\mathbf{u}} \rangle) \cdot \nabla_\rho \bar{h} \, \mathrm{d}\rho, \tag{14b}$$

$$\phi_2 = 2\sigma_t^{3/2} \left(1 + \sigma_t^{-1}\right) f^{-1} r_d^{-2} \int_{\rho_s}^{\rho_b} |\bar{\mathbf{u}} - \langle \bar{\mathbf{u}} \rangle|^2 \Gamma^{1/2}(\rho) \bar{h} \, \mathrm{d}\rho. \tag{14c}$$

In addition, we have

$$\phi_1 > 0, \quad \phi_2 < 0. \tag{14d}$$

While the second inequality is obvious, the first one follows from (I.45b,c) since $\phi_1 \sim -I_{3,4}$ (I.45b). Result (14) has a simple interpretation: ϕ_1 represents a positive contribution to the eddy production due to the baroclinic instability while ϕ_2 represents an eroding action of mean velocity on the coherent structure of eddies since the eddy drift velocity differs from the mean velocity field (compare with the discussion of the approximate formula (I.44a)). Even though result (11a) is almost model independent (it requires only the horizontal locality of eddy production), the representation (14) and its interpretation is in the framework of the present model.

2.3.2 Dissipation of eddy energy. As shown in I, the release of eddy potential energy occurs at the scales which correspond to the spectral Rossby number of the order 1. At these scales, as it follows from equation (I.38c), $\mathrm{Ro}(q)$ relates to the spectral Richardson number

$$\mathrm{Ri}(q) \equiv N^2/S^2(q) = C_R \mathrm{Ro}^{-1}(q), \quad C_R \sim 1. \tag{15a}$$

Therefore, in the considered region $\mathrm{Ri} \sim 1$ that is the condition to generate the diapycnal turbulence which leads to the eddy energy dissipation ϵ_d. Below we show that (15a) together with results (I.38a,b) allows us to parameterize ϵ_d with a globally invariant factor. To simplify the further analysis, as a first step we idealize the process of releasing eddy potential energy by assuming that it occurs at a fixed wave number $\tilde{q}$. Introducing the notations $\tilde{\mathrm{Ro}} \equiv \mathrm{Ro}(\tilde{q})$, $\tilde{S}^2 \equiv S^2(\tilde{q})$ and $\tilde{\mathrm{Ri}} \equiv \mathrm{Ri}(\tilde{q})$, we can represent

$$\epsilon_d = C_S^2 F(\tilde{\mathrm{Ri}}) \tilde{S}^3 \tilde{z}'^2, \tag{15b}$$

where $C_S \sim 0.15$ is the Smagorinsky–Lilly constant which we separate so to normalize the monotonic function $F(x)$ as follows:

$$F(0) = 1, \quad F(x) = 0 \quad \text{for} \quad x > \mathrm{Ri}_{cr}, \quad \mathrm{Ri}_{cr} \approx 1. \tag{15c}$$

As for the vertical scale of the diapycnal turbulence, $\tilde{z}'$, it is deduced from results (I.38a,b) and (15a)

$$\tilde{z}' = C_v f \tilde{\mathrm{R}}\mathrm{i}^{-1/2}(N\tilde{q})^{-1}, \quad C_v \sim 1. \tag{15d}$$

It is convenient to transform (15b) into the form

$$\epsilon_d = C_S^2 \tilde{\mathrm{R}}\mathrm{i}^{-3/2} F(\tilde{\mathrm{R}}\mathrm{i}) N^3 \tilde{z}'^2. \tag{15e}$$

We can decrease the number of unknown variables by expressing $\tilde{q}$ in terms of $\tilde{\mathrm{R}}\mathrm{i}$ with use of (15a) and (I.37c,39c)[4]

$$\tilde{q}^2 = C_q r_d f^3 K^{-3/2} \tilde{\mathrm{R}}\mathrm{i}^{-3}, \qquad C_q \sim 1. \tag{15f}$$

Finally, from (15d–f) we deduce

$$\epsilon_d = A_d N (f r_d)^{-1} K^{3/2}, \tag{16a}$$

where

$$A_d = C_S^2 a_d, \quad a_d = C_v^2 C_q^{-1} \tilde{\mathrm{R}}\mathrm{i}^{1/2} F(\tilde{\mathrm{R}}\mathrm{i}). \tag{16b}$$

Because of the monotony of $F(x)$ and condition (15c), $\mathrm{Ri}^{1/2}F(\mathrm{Ri}) < 1$. Then it is convenient to rewrite (16b) as follows:

$$A_d = 10^{-2} C_d^{-1}, \quad C_d \leq 1. \tag{16c}$$

To evaluate a typical ϵ_d, we substitute into (16a) $N \sim 3 10^{-3}\,\mathrm{s}^{-1}$, $r_d \sim 3 10^4\,\mathrm{m}$ and $K \sim 2 10^{-3}\,\mathrm{m}^2\mathrm{s}^{-2}$ that corresponds the typical eddy energy $\sim 10^{-2}\,\mathrm{m}^2\mathrm{s}^{-2}$ and the filling factor of the order a few tens percents. Then we obtain $\epsilon_d \sim 10^{-9}\,\mathrm{m}^2\mathrm{s}^{-3}$ that is in accordance with the evaluation of the eddy production performed below (13e), and with direct measurements (Gargett *et al.*, 1981).

Really the Ro-region where releasing eddy potential energy occurs, is spread around some $\tilde{\mathrm{R}}\mathrm{o} \sim 1$ that results in substituting $\tilde{\mathrm{R}}\mathrm{i}^{1/2} F(\tilde{\mathrm{R}}\mathrm{i}) \to \int \mathrm{Ri}^{1/2} F(\mathrm{Ri}) \mathrm{P}(\mathrm{Ri}) \mathrm{d}(\mathrm{Ri})$ in (16b) where $\mathrm{P}(x)$ is some probability distribution function. We stress that all constants in equations (15,16) are flow independent, at least, approximately. One of consequences of the independence is the approximate universality of the flux Richardson number R_f and related to it the ratio $\gamma \equiv \epsilon_p/\epsilon_v = R_f(1 - R_f)^{-1}$ which is thought to be about 0.2 (Wunsch and Ferrari, 2004), where ϵ_p is the fraction of the dissipation available to vertically mix the density field and ϵ_v is the viscous dissipation.

[4]There is a misprint in (I.37c) where ε has to be in the power 1/3 instead of the written one 1/2.

2.3.3 Surface eddy kinetic energy in terms of large scale fields. Since the eddy production is almost horizontally local, we can equate the column eddy energy production, equation (11a), to the column eddy energy dissipation which is obtained by integrating expression (16a) over the depth, i.e. multiplying (16a) by $-\bar{h}\mathrm{d}\rho$. Since in any case the locality of the production is approximate, we use approximate relation (13d) in (11a) that leads to expressions (14a–c) for the production. Substituting also definition (1g) into (16a), we obtain the following result for the surface eddy energy

$$K_s = C_d 10^2 (fr_d)^2 (\phi_1 + \phi_2)/\phi_3, \tag{17a}$$

where $\phi_{1,2}$ are given in (14b,c) while

$$\phi_3 = -\int_{\rho_s}^{\rho_b} N\Gamma^{3/2}(\rho)\bar{h}\mathrm{d}\rho. \tag{17b}$$

As we discuss in section 4.6 of I, result (17a) which is close to result (I.48b), is valid only if it yields $K_s > 0$, otherwise the model leads to $K_s = 0$. It is worth noticing that (17a) is rather close to result (I.47) with the two important differences: (1) (17a) is based on the more accurate and model independent expression for the production, equation (11a), in comparison with analogous relation (I.44a); (2) The unknown constant $C_d \sim 1$ in (17a), as discussed above, does not depend on large scale fields, i.e. is globally invariant while the constant $\tilde{F}_e$ in (I.47a,48b) which is defined in (I.46e), relates to C_d as follows:

$$\tilde{F}_e = 10^2 C_d \left[\int_{\rho_s}^{\rho_b} (N/f)\Gamma^{3/2}(\rho)\bar{h}\,\mathrm{d}\rho\right]^{-1} \int_{\rho_s}^{\rho_b} \Gamma^{3/2}(\rho)\bar{h}\,\mathrm{d}\rho \tag{17c}$$

and thus depends on the stratification. This relation is deduced by equating the column dissipation obtained from equations (I.46a,b, 39c) on the one hand, and from equation (16a) on the other.

To compute K_s with use of equation (17a), we need the normalized kinetic energy profile $\Gamma(\rho)$, (1g). In the zeroth order of the parameters $\Omega_{1,2}\tau$, we have (see(I.48a))

$$\Gamma_0 = B_1^2(\rho) \quad \text{with} \quad B_1(\rho_s) = 1 \tag{18a}$$

where $B_1(\rho)$ is the eigenfunction of (I.25d) with the boundary conditions (I.19a,d). To find more accurate value of K_s with use of (17a), we need the next approximations for Γ which we find in the next subsection more accurately than in paper I.

2.3.4 Normalized eddy kinetic energy profile in terms of large scale fields. As we discuss in Appendix A, the correction $\Delta\Gamma \equiv \Gamma(\rho) - \Gamma_0(\rho)$ to the zeroth approximation (18a) is of the second order of $\Omega_{1,2}\tau$ and is contributed mainly by the barotropic component to be

$$\Delta\Gamma = [I \cdot B_1^2(\rho)]\left[\left(\frac{1}{2} + c_1\right)\mathbf{I}_1^2 + \mathbf{I}_1 \cdot \mathbf{I}_2 + \mathrm{trace}(\mathbf{T}_1 + \mathbf{T}_2)\right]/K_s \tag{18b}$$

where $\mathbf{I}_{1,2}, \mathbf{T}_{1,2}, c_{1,2}$ are given in (4b–f).

Equations (18a,b), (17a,b), (14b,c) are the system for determining $\Gamma(\rho)$ and K_s which can be solved perturbatively. As we have mentioned just above, the zeroth order solution for K_s can be found from equation (17a,b), (14b,c) by substituting there the zeroth order approximation (18a) for Γ. To compute corrections to Γ_0, in the right hand side of (18b) (with account for (4b–f)) we may substitute Γ_0 and the found value of the zeroth approximation for K_s. This procedure has to be slightly modified when one computes $\mathbf{I}_2$, (4b), since in this case integral (4f) diverges in the approximation (18a) in the region where $B_1(\rho) \to 0$. To overcome this problem, one may substitute $B_1(\rho) \to \alpha$ in the region where $|B_1(\rho)| < \alpha$ and choose, say, $\alpha = 0.2$ or even more.

3. Eddy parameterization in level coordinates

The problem of the transformation of subgrid terms from isopycnal to level coordinates is, generally speaking, rather complex due to the random nature of the density field. However, due to condition (8b), the coordinates transformation problem can be solved perturbatively as an expansion in powers of $h'/\bar{h}$. The complete solution of this problem is presented in the works by McDougall (1998) and McDougall and McIntosh (2001). Using the transformation formulae, one can express sub-grid eddy second order moments which are present in equations for large scale fields in level coordinates, in terms of analogous moments in isopycnal coordinates. The latter, in turn, are computed in the framework of the present eddy dynamic model. Below this method of eddy parameterizing in level coordinates is applied to the equations for large scale density and momentum.

3.1 *Eddy parameterization in the large scale density equation*

We begin with the large scale density equation in the Boussinesq approximation

$$\frac{\partial\overline{\overline{\rho}}}{\partial t} + \overline{\overline{\mathbf{u}}}\cdot\nabla_H\,\overline{\overline{\rho}} + \frac{\partial\overline{\overline{\rho}}}{\partial z}\overline{\overline{w}} + \nabla_H\cdot\mathbf{F}_M^H + \frac{\partial}{\partial z}F_M^V + G = 0, \tag{19a}$$

where

$$\mathbf{F}_M^H = \overline{\overline{\mathbf{u}''\rho''}} \quad \text{and} \quad F_M^V = \overline{\overline{w''\rho''}} \tag{19b, c}$$

are the horizontal and vertical components of the $3D$-sub-grid mass flux; $\mathbf{u}, w$ are analogous components of $3D$ velocity $\mathbf{U}$, $\overline{\overline{A}}$ denotes an average value while $''$ marks fluctuating fields in level coordinates, ∇_H is the gradient operators in 2D-space at a fixed z. G is the diabatic term. In coarse resolution OGCM's equation (19a) is usually considered in the following equivalent form (Andrews and McIntyre, 1976; see also Treguier *et al.*, 1997):

$$\frac{\partial\overline{\overline{\rho}}}{\partial t} + (\overline{\overline{\mathbf{u}}} + \mathbf{u}^+)\cdot\nabla_H\,\overline{\overline{\rho}} + (\overline{\overline{w}} + w^+)\frac{\partial\overline{\overline{\rho}}}{\partial z} + \Sigma_z + G = 0, \tag{19d}$$

where $(\mathbf{u}^+, w^+)$ is the 3D eddy induced velocity (see Gent *et al.*, 1995) which is an analog of the bolus velocity $\mathbf{u}_*$ and which is related to the fluctuating fields via

$$\mathbf{u}^+ = -\frac{\partial}{\partial z}\left[\left(\frac{\partial\overline{\overline{\rho}}}{\partial z}\right)^{-1}\overline{\overline{\mathbf{u}''\rho''}}\right], \tag{19e}$$

$$w^+ = -\int_{z_b}^{z} \nabla_H \cdot \mathbf{u}^+(\mathbf{r},\xi)\,\mathrm{d}\xi = \int_{z}^{0} \nabla_H \cdot \mathbf{u}^+(\mathbf{r},\xi)\,\mathrm{d}\xi. \tag{19f}$$

Finally

$$\Sigma = F_M^V + \left(\frac{\partial\overline{\overline{\rho}}}{\partial z}\right)^{-1} \mathbf{F}_M^H \cdot \nabla_H \overline{\overline{\rho}}, \tag{19g}$$

which is nothing but a residual density flux across **mean** isopycnals. Even though all fluxes across true isopycnals vanish in the adiabatic approximation, this is not the case for Σ (see details in Dubovikov and Canuto, 2005). Thus, one can see that the eddy parameterization problem in density equation (19d) consists of two parts: (1) modelling $(\mathbf{u}^+, w^+)$; (2) modelling Σ. The former has a vast literature starting from the pioneer work by Gent and McWilliams (1990) whereas the second problem is rarely discussed. Instead, it has been assumed that Σ is negligible. However, a fine resolution simulation by Gille and Davis (1999) concluded that the Σ-term is important. This conclusion is consistent with the present model which is capable to compute Σ (see section 3.1.2).

3.1.1 Parameterization of the eddy induced velocity. In the present work we restrict ourselves by the lowest order of $h'/\overline{h}$ in which the transformation formulae are simplified to yield

$$\frac{\partial\overline{\overline{\rho}}}{\partial z} = \bar{h}^{-1}, \quad \left(\frac{\partial\overline{\overline{\rho}}}{\partial z}\right)^{-1}\rho'' = -z', \quad \mathbf{u}'' = \mathbf{u}', \quad \overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}}. \tag{20a}$$

McDougall (1998) and McDougall and McIntosh (2001) compute corrections to the approximation of (20a) which we do not account for in the present work. Within the considered approximation, equation (19e) can be rewritten as

$$\mathbf{u}^+ = \overline{h}^{-1}\frac{\partial}{\partial\rho}\left(\overline{z'\mathbf{u}'}\right), \tag{20b}$$

i.e. $\mathbf{u}^+$ coincides with the eddy production velocity $\mathbf{u}_p$ introduced in (11b). Within the same accuracy, approximation (13d) is valid and thus we may adopt

$$\mathbf{u}^+ = \mathbf{u}_*. \tag{20c}$$

Corrections to this relation are given in the work of McDougal and McIntosh (2001). What remains, is just to express $\mathbf{u}^{+}$and κ_h in terms of level coordinates. With use of (20a), (1h–j) we obtain the results below.

Eddy induced velocity:

$$\mathbf{u}^{+} = -\kappa_h \mathbf{S}, \tag{21a}$$

$$\mathbf{S} = \frac{\partial \mathbf{L}}{\partial z} - \left\langle \frac{\partial \mathbf{L}}{\partial z} \right\rangle + \left[1 + \sigma_t^{-1}\right] f^{-1} r_d^{-2} \mathbf{e}_z \times \left(\overline{\overline{\mathbf{u}}} - \langle \overline{\overline{\mathbf{u}}} \rangle\right), \tag{21b}$$

$$\mathbf{L} = -\left(\frac{\partial \overline{\overline{\rho}}}{\partial z}\right)^{-1} \nabla_H \overline{\overline{\rho}}, \tag{21c}$$

$$\langle A \rangle \equiv \left[\int_{z_b}^{z_s} \Gamma^{1/2}(z)\,\mathrm{d}z\right]^{-1} \int_{z_b}^{z_s} A(z)\Gamma^{1/2}(z)\,\mathrm{d}z, \tag{21d}$$

$$\Gamma(z) \equiv \Gamma[\rho(z)]. \tag{21e}$$

Using the same symbol $\mathbf{S}$ in equations (1h) and (21a) stresses the equivalence of expressions (1i) and (21b) in the lowest order of $h'/\overline{h}$.

The mesoscale diffusivity in (21a) may be derived from equation (1j):

$$\kappa_h = 2\sigma_t^{3/2} r_d K(z)^{1/2}, \tag{21f}$$

where

$$K(z) = K_s \Gamma(z) \tag{21g}$$

and K_s is computed with use of equation (17a) together with

$$\phi_1 = -2\sigma_t^{3/2} \int_{-H}^{0} \Gamma^{1/2}(z) \mathbf{e}_z \times (\bar{\mathbf{u}} - \langle \bar{\mathbf{u}} \rangle) \cdot \frac{\partial \mathbf{L}}{\partial_z}\,\mathrm{d}z, \tag{21h}$$

$$\phi_2 = -2\sigma_t^{3/2}\left(1 + \sigma_t^{-1}\right) f^{-1} r_d^{-2} \int_{-H}^{0} |\bar{\mathbf{u}} - \langle \bar{\mathbf{u}} \rangle|^2 \Gamma^{1/2}(z)\,\mathrm{d}z, \tag{21i}$$

$$\phi_3 = \int_{-H}^{0} N \Gamma^{3/2}(z)\,\mathrm{d}z. \tag{21j}$$

These relations are equivalent with the adopted accuracy of (14b,c), (17b) in isopycnal coordinates. In the zeroth approximation for K_s, in equations (21h–j) we may use the approximation

$$\Gamma(z) \approx \Gamma_0(z) = |B_1(z)|^2 \equiv |B_1[\rho(z)]|^2 \tag{22a}$$

with the normalization $B_1(0) = 1$. To find the correction $\Delta\Gamma \equiv \Gamma(z) - \Gamma_0(z)$, one may use relation (18b) together with equations (4a–e) in which $\mathbf{S}$ is defined in (21b) and

$$\mathbf{u}_d = \langle\overline{\mathbf{u}}\rangle + (1+\sigma_t)^{-1}\mathbf{c} - f\,r_d^2\left(1+\sigma_t^{-1}\right)^{-1}\mathbf{e}_z \times \left\langle\frac{\partial \mathbf{L}}{\partial z}\right\rangle \tag{22b}$$

and the averaging operations $\langle X\rangle$ and $\langle X\rangle_0$ are defined for any function $X(z)$ in (21d) and as

$$\langle X\rangle_0 \equiv H^{-1}\int_{-H}^{0} X(z)\,\mathrm{d}z. \tag{22c}$$

As well as in isopycnal coordinates, result (17a) is valid only if it yields $K_s > 0$, otherwise the model leads to $K_s = 0$.

The vertical component of the eddy induced velocity is found from one of relations (19f) which are consistent only if $\langle \mathbf{u}_*\rangle_0 = \mathbf{0}$, i.e. $\mathbf{u}_*$ is pure baroclinic. Result (21a,b,f) does satisfy this condition in accordance with the common expectation and thus (19f) yields the condition

$$w^+(0) = w^+(-H) = 0 \tag{22d}$$

which is also expected.

3.1.2 Computing the Σ-term. Using the transformation relations (20a) and

$$\nabla_H = \nabla_\rho + (\nabla_H \rho)\frac{\partial}{\partial\rho}, \qquad \frac{\partial}{\partial z} = \frac{\partial\rho}{\partial z}\frac{\partial}{\partial\rho}, \tag{23a}$$

one may show that within the considered approximation which is of the main order in the small parameter (8b), the right hand side of equation (7e) coincides with Σ, equation (19g), multiplied by $g\rho^{-1}$, i.e.

$$\frac{\partial W}{\partial t} = g\rho^{-1}\Sigma. \tag{23b}$$

The same result may be deduced from the equation for the variance of density fluctuation in the adiabatic approximation

$$\frac{\partial}{\partial t}\left(\overline{\overline{\rho''^2}}\right) = -2\frac{\partial\overline{\overline{\rho}}}{\partial z}\Sigma - \overline{\overline{\mathbf{U}}}\cdot\nabla\overline{\overline{\rho''^2}} - \nabla\cdot\left(\overline{\overline{\mathbf{U}''\rho''^2}}\right), \tag{23c}$$

where $\mathbf{U}$ is $3D$ velocity $(\mathbf{u}, w)$. It is usually suggested (see, e.g., Treguier *et al.*, 1997) to consider a stationary flow, i.e. to neglect the left hand side of (23c). However, such an analysis is not applicable in the adiabatic approximation. In fact, in the case of stationary flows in the complete density equation for $\partial(\overline{\overline{\rho''2}})/\partial t$ the production term $-2\partial\overline{\overline{\rho}}/\partial z\Sigma$ is mainly balanced by the dissipation due to the diapycnal mixing. In the adiabatic approximation the dissipation is absent. Then, apart from the negligible diffusion and advection terms (see Appendix C), the production term may be balanced only by the adiabatic growth rate of density variance which because of $\overline{\rho'^2} = 2N^2 W$ is proportional to the adiabatic growth rate of eddy potential energy $\partial W/\partial t$. Thus, again we arrive at equation (23b). To evaluate Σ, we may adopt $\partial W/\partial t \sim \epsilon$ which, using the results by Gargett *et al.*, (1981), is estimated to be $(10^{-8} - 10^{-9})\,\mathrm{m^2s^{-3}}$. Therefore, from (23b) we get $\Sigma \sim (10^{-6} - 10^{-7})\,\mathrm{kgm^{-2}s^{-1}}$ and therefore $\partial\Sigma/\partial z \sim 10^{-9} - 10^{-10}\,\mathrm{kgm^{-3}s^{-1}}$. This result should be compared with the contribution to (19d) of the terms containing $\mathbf{u}^+, w^+$:

$$\mathbf{u}^+ \cdot \nabla_H \rho \sim w^+ \frac{\partial \rho}{\partial z} \sim u^+ H L^{-1} \frac{\partial \rho}{\partial z} \sim 10^{-9}\mathrm{kgm^{-3}s^{-1}}. \tag{23d}$$

The last result is obtained using the typical values $\partial\rho/\partial z \sim 10^{-3}\,\mathrm{kgm^{-4}}$ and $u^+ \sim 10^{-3}\,\mathrm{ms^{-1}}$.

Thus, the contribution of the Σ-term into equation (19d) must be accounted for. To express Σ via (23b) in terms of large scale fields, we need to solve the analogous problem for $\partial W/\partial t$. To solve the latter, we use the fact that the characteristic time of eddy energy production which is $L/|\overline{\mathbf{u}}| \sim 10^8$s, is much longer than the time to achieve the virial relation $\mathcal{W} = \sigma_t \mathcal{K}$ between column eddy potential and kinetic energy (equation I.42d). The latter time is of the order of $r_d/u' \sim (10^5 - 10^6)$s. This implies that the vertical profiles of $\partial W/\partial t$ and W are proportional to each other. Thus, we may write

$$\frac{W_t}{(\partial\mathcal{W}/\partial t) + (\partial\mathcal{K}/\partial t)} = \frac{W}{\mathcal{W} + \mathcal{K}} = \frac{1}{1 + \sigma_t^{-1}} \frac{W}{\mathcal{W}}. \tag{23e}$$

In the adiabatic approximation, the denominator in the left hand side equals the column eddy energy production $\mathcal{P}_e$ given in equations (14a),(21h,i). Expression (13e) for W in z-coordinates transforms into

$$W = \sigma_t \big(f r_d N^{-1} \big)^2 \left(\frac{\partial B_1}{\partial z} \right)^2 K_s. \tag{23f}$$

Now we substitute this expression, together with (14a) and (21h,i), into (23e) and further into (23b) and take into account the relation $\mathcal{W} = W\,\mathrm{d}z$. After simple manipulations with account for the boundary conditions (I.19a,d), we obtain

$$\Sigma = 10 C_d^{1/2} (1 + \sigma_t^{-1})^{-1} \rho r_d^4 g^{-1} f^4 N^{-2} \left[\frac{\partial}{\partial z} B_1(z) \right]^2 (\phi_1 + \phi_2)^{3/2} \phi_3^{-1/2} \Psi^{-1}, \tag{23g}$$

where $\phi_{1,2,3}$ are given in equations (21h–j) and

$$\Psi = \int_{z_b}^{z_s} B_1^2\,\mathrm{d}z. \tag{23h}$$

In (23g,h) the eigenfunction $B_1(z)$ is normalized $B_1(0) = 1$. For numerical evaluations it is useful to express result (23g) in terms of the surface eddy kinetic energy K_s:

$$\Sigma = 0.1C_d^{-1/2}(1+\sigma_t^{-1})^{-1}\rho f r_d g^{-1} N^{-2}\left[\frac{\partial}{\partial z}B_1(z)\right]^2 K_s^{3/2}\phi_3\Psi^{-1}. \tag{23i}$$

Substituting here $C_d \sim 1$, $K_s \sim 210^{-3}\mathrm{m}^2\mathrm{s}^{-2}$, $\partial B_1/\partial z \sim H^{-1}$, $\phi_3 \sim NH$, $\Psi \sim H$ and the standard values of ρ, r_d, g, f, N, we get, as above, $\Sigma \sim 10^{-6}\mathrm{kgm}^{-2}\mathrm{s}^{-1}$ and therefore $\Sigma_z \sim 10^{-9}\mathrm{kgm}^{-3}\mathrm{s}^{-1}$. Since the last result is of order (23d), we conclude that in density equation (19d) the term Σ_z cannot be neglected and its parameterization in terms of large scale fields is given in equation (23g).

Notice that due to $\partial B_1/\partial z = 0$ at the surface and bottom, expressions (23g,i) satisfy the boundary condition

$$\Sigma(0) = \Sigma(H) = 0, \tag{23j}$$

as required for any model for Σ.

3.2 *Eddy parameterization in mean momentum equation*

Next, consider the large scale horizontal momentum equation in level coordinates (without sources)

$$\frac{\partial\overline{\overline{\mathbf{u}}}}{\partial t} + \overline{\overline{\mathbf{u}}}\cdot\nabla_H\overline{\overline{\mathbf{u}}} + \overline{\overline{w}}\frac{\partial\overline{\overline{\mathbf{u}}}}{\partial z} + f\mathbf{e}_z\times\overline{\overline{\mathbf{u}}} = -\rho_0^{-1}\nabla_H\overline{\overline{p}} - \mathbf{A}^a - \mathbf{A}^d, \tag{24a}$$

where $\mathbf{A}^a$ denotes the contribution of adiabatic sub-grid mixing due to eddies fields and $\mathbf{A}^d$ corresponds to a diabatic sub-grid mixing. As well as in isopycnal coordinates, the developed eddy model considers only $\mathbf{A}^a$ which further we decompose into the horizontal and vertical mixing components

$$\mathbf{A}^a = \mathbf{A}_H + \mathbf{A}_V, \quad \mathbf{A}_H = \overline{\overline{\mathbf{u}''\cdot\nabla_H\mathbf{u}''}}. \quad \mathbf{A}_V = \overline{\overline{w''\frac{\partial\mathbf{u}''}{\partial z}}} \tag{24b, c, d}$$

where the fluctuating fields $\mathbf{u}'', w''$ include only the contributions of horizontal scales $\sim r_d$, that is, only the adiabatic contribution.

3.2.1 Computation of $\mathbf{A_H}$.

Instead of expression (24c), $\mathbf{A}_H$ may be presented in the following equivalent form

$$\mathbf{A}_H = \mathbf{e}_z\times\mathbf{F}_{\zeta''} + \nabla_H K, \quad \mathbf{F}_{\zeta''} = \overline{\overline{\zeta''\mathbf{u}''}}, \quad \zeta'' = \nabla_H\times\mathbf{u}''\cdot\mathbf{e}_z \tag{24e, f, g}$$

(compare with equations (2d),(3a) in isopycnal coordinates). In analogy with the methodology in isopycnal coordinates, we split $\mathbf{F}_{\zeta''}$ into the geostrophic and ageostrophic components

$$\mathbf{F}_{\zeta''} = \mathbf{F}^g + \mathbf{F}^{ag}, \qquad \mathbf{F}^g = \overline{\overline{\zeta''\mathbf{u}''_g}}, \qquad \mathbf{F}^{ag} = \overline{\overline{\zeta''\mathbf{u}''_a}}. \tag{25a}$$

Then $\mathbf{A}_H$ is split as follows:

$$\mathbf{A}_H = \mathbf{A}^g + \mathbf{A}^{ag}, \qquad \mathbf{A}^g = \nabla_H K + \mathbf{e}_z \times \mathbf{F}^g, \qquad \mathbf{A}^{ag} = \mathbf{e}_z \times \mathbf{F}^{ag}. \tag{25b}$$

In full analogy with isopycnal coordinates, since $\mathbf{u}''_g$ is divergence free in XY-plane and ζ'' (24g) is contributed only by $\mathbf{u}''_g$, one can derive from (25a) that

$$\mathbf{e}_z \times \mathbf{F}^g = \nabla_H \cdot [\mathbf{R}^g], \tag{25c}$$

where $[\mathbf{R}^g]$ is the traceless component of the two dimensional geostrophic Reynolds stress in XY-plane:

$$\mathbf{R}^g = \overline{\overline{\mathbf{u}''_g \mathbf{u}''_g}}. \tag{25d}$$

In (25c) we adopt the notation (3f) with the substitution $\boldsymbol{\delta}_\rho \to \boldsymbol{\delta}_H$ where $\boldsymbol{\delta}_H$ is $2D$ Kroneker tensor in a horizontal plane. In the approximation (20a), from (3e) and (25d) we have

$$[\mathbf{R}^g] = [\mathbf{R}_g] \tag{25e}$$

and thus we may use result (4a–e) together with the definition (22c). Finally, substituting (25c) into (25b) and taking into account that

$$\mathbf{R}^g = \boldsymbol{\delta}_H K + [\mathbf{R}^g], \tag{25f}$$

where $\boldsymbol{\delta}_H$ is the two-dimensional Kroneker tensor in the horizontal plane, we get

$$\mathbf{A}^g = \nabla_H \cdot \mathbf{R}^g. \tag{25g}$$

As for as $\mathbf{A}^{ag}$, it can be obtain by applying relations (20a) and (23a) to (25a,b), (24g). Indeed,

$$\mathbf{F}^{ag} = \overline{\overline{\mathbf{u}''_a \nabla_H \times \mathbf{u}'' \cdot \mathbf{e}_z}} = \overline{\mathbf{u}'_a \nabla_\rho \times \mathbf{u}' \cdot \mathbf{e}_\rho} + \overline{\overline{\mathbf{u}''_a (\nabla_H \rho) \times \left(\frac{\partial \mathbf{u}''}{\partial z}\right) \cdot \left(\frac{\partial \rho}{\partial z}\right)^{-1}}}. \tag{26a}$$

Because of (8b), we may substitute $\partial\rho/\partial z \to \partial\overline{\overline{\rho}}/\partial z$ in the second term of the right hand side. In addition, we may substitute here $\nabla_H \rho \to \nabla_H \overline{\overline{\rho}}$ that implies neglecting the triple correlation. Thus, in the second term, the operator ∇_H acts on the mean field $\overline{\overline{\rho}}$ whose characteristic length scale $L \sim 10^6\,\mathrm{m}$. Therefore, here $\nabla_H \sim 10^{-6}\,\mathrm{m}^{-1}$ whereas in the first term $\nabla_\rho \sim r_d^{-1} \sim 3 \times 10^{-5}\,\mathrm{m}^{-1}$ since it acts on the eddy field $\mathbf{u}'$. Therefore, the second term may be neglected in comparison with the first one. Then from (26a) and the last relation (3b) we get

$$\mathbf{F}^{ag} = \mathbf{F}_{ag}. \tag{26b}$$

Substituting here result (6c), with account for (20a) we get

$$\mathbf{F}^{ag} = \left[\beta f^{-1}\mathbf{e}_x - (\sigma_t r_d^2 f)^{-1}(\bar{\bar{\mathbf{u}}} - \mathbf{u}_d)\right]K, \tag{27}$$

where $\mathbf{u}_d$ is given in (22b).

3.2.2 Computation of $\mathbf{A_V}$. As in the case of $\mathbf{A}_H$, in definition (24d) we express the fluctuating fields in level coordinates in terms of an isopycnal system where eddy dynamic equations have been developed. We begin with the field w'' which in approximation (20a) equals w' given in equation (7c). We write the latter equation in Fourier-space $\omega, \mathbf{q}$ taking into account that its non-linear term (in parenthesis) in the framework of the present model, may be deduced by integrating the non-linear term $-\tilde{\chi}h'$ of the equation for h' (I.14b) over ρ. Accounting for also equation (1d) and boundary condition (I.19a,d), we obtain

$$w''(\mathbf{q}) = \mathrm{i}\mathbf{q}\cdot(\bar{\mathbf{u}} - \mathbf{u}_d)z'(\mathbf{q}) + \mathbf{u}'(\mathbf{q})\cdot\nabla_\rho\bar{z} + \tilde{\chi}z'(\mathbf{q}) - \int_{\rho_s}^{\rho}\frac{\partial\tilde{\chi}}{\partial\rho}\,z'(\mathbf{q})\,\mathrm{d}\rho. \tag{28}$$

Now we express the second field $\partial\mathbf{u}''/\partial z$ coming into definition (24d), in terms of fields in isopycnal system. To this end, we make use of equation (20a) and the geostrophic relation in isopycnal coordinates in Fourier-space

$$\frac{\partial\mathbf{u}''}{\partial z}(\mathbf{q}) = \bar{\bar{\rho}}_z\frac{\partial\mathbf{u}'}{\partial\rho}(\mathbf{q}) = -\mathrm{i}N^2 f^{-1}\mathbf{e}_z\times\mathbf{q}z'(\mathbf{q}). \tag{29}$$

Thus, in equations (28),(29) we have expressed the ingredients of the sub-grid term $\mathbf{A}_V$, equation (24d), in terms of the fluctuating fields in isopycnal coordinates and we are ready to compute $\mathbf{A}_V$ in Fourier space:

$$\delta(\mathbf{q} - \mathbf{q}')\tilde{\mathbf{A}}_V(\mathbf{q}) = \mathrm{Re}\left\{\overline{w''(\mathbf{q})\frac{\partial\mathbf{u}''^*}{\partial z}(\mathbf{q}')}\right\}. \tag{30a}$$

To obtain the spectrum, one needs to integrate (30a) over $\mathbf{q}'$ and $\mathbf{n}$. In accordance with the general definition of spectra (5g), we obtain

$$\mathbf{A}_V(q) = q\int\tilde{\mathbf{A}}_V(\mathbf{q})\,\mathrm{d}\mathbf{n}, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|. \tag{30b}$$

Substituting (28) and (29) into (30a), we notice that the contributions due to the last two terms in the right hand side of (28) equal zero because of the factor i in front of the last expression in (29). The other two terms of (28) with account for (7a), (3e) yield the following result

$$\mathbf{A}_V(q) = q^2 f^{-1}(\mathbf{u} - \mathbf{u}_d)\times\mathbf{e}_z W(q) + (\nabla_\rho\bar{z})\cdot\frac{\partial\mathbf{R}_g}{\partial z}(q), \tag{30c}$$

where $W(q)$, $\mathbf{R}_g$ (9) are the spectra of eddy potential energy and the geostrophic Reynolds stress defined in (3e). In carring out the integration in (30b), we have taken into account relation (6b). The integration (30c) over q reduces to the substitution of the spectra with the corresponding functions. In addition, we express $\nabla_\rho \bar{z}$ in terms of level coordinates and use (25e). The result is

$$\mathbf{A}_V = -(\sigma_t r_d^2 f)^{-1} W \mathbf{e}_z \times (\bar{\bar{\mathbf{u}}} - \mathbf{u}_d) - \left[\left(\frac{\partial \bar{\bar{\rho}}}{\partial z} \right)^{-1} \nabla_H \bar{\bar{\rho}} \right] \cdot \frac{\partial \mathbf{R}^g}{\partial z}. \tag{31}$$

3.2.3 Total adiabatic eddy contribution to the mean momentum equations. Substituting (25b,g), (27) and (31) into (24b), we obtain the total adiabatic eddy contribution to the mean momentum equations in level coordinates (24a)

$$\mathbf{A}^a = \nabla_{\bar{\bar{\rho}}} \cdot \mathbf{R}^g + \left(\sigma_t r_d^2 f \right)^{-1} E_{\text{tot}} (\bar{\bar{\mathbf{u}}} - \mathbf{u}_d) \times \mathbf{e}_z + \beta f^{-1} K \mathbf{e}_y, \tag{32a}$$

where E_{tot}, as well as in (10a), is the total eddy energy, $\nabla_{\bar{\bar{\rho}}}$ is the gradient operator within a mean isopycnal surface

$$\nabla_{\bar{\bar{\rho}}} = \nabla_H - \left(\frac{\partial \bar{\bar{\rho}}}{\partial z} \right)^{-1} \nabla_H \bar{\bar{\rho}} \frac{\partial}{\partial_z}. \tag{32b}$$

With account for (21g), (22a), (23f) in the zeroth order of $\Omega_{1,2}\tau$,

$$E_{\text{tot}} \equiv K + W = \left[B_1^2 + \sigma_t (r_d f)^2 N^{-2} \left(\frac{\partial B_1}{\partial z} \right)^2 \right] K_s. \tag{32c}$$

Substituting a typical value of large scales $L \sim 10^6$ m and $E_{\text{tot}} \sim K \sim |\mathbf{R}^g| \sim (10^{-2} - 10^{-3})\,\text{m}^2\text{s}^{-2}$ into (32a), we conclude that the first term in the right hand side dominates and is of order $(10^{-8} - 10^{-9})\,\text{ms}^{-2}$, the second term is smaller almost by order of magnitude, while the third term is smaller again by order of magnitude.

4. Conclusion

In the present work we have applied the dynamic mesoscale eddy model developed in I, II, to compute the eddies contributions to large scale equations. In isopycnal coordinates, we compute the eddy contribution to the adiabatic sub-grid mixing $\mathbf{A}^a$ which appears in the mean momentum equations (2c) and contains the gradient of the Reynolds stress together with the a-geostrophic component of the eddy vorticity flux; equations (6d), (3g), (4a–f), (1g), (17a,b), (18a,b). The eddy contribution to the large scale momentum equation differs from the traditional down-gradient parameterization. To compute the eddy contributions to the large scale equations in level coordinates, one needs to transform the eddy fields from isopycnal coordinates. On the basis of the transformation formulae and the developed eddy dynamics in isopycnal coordinates, we have computed the eddy contributions to large scale equations in level coordinates. In particular, we have found that the contribution of the residual density flux across mean isopycnals into the density equation has the same order as the contributions of the terms with the eddy induced velocity.

Two major topics remain to be studied. First, the mesoscale parameterization must be completed by the parameterizations in the temperature and salinity equations. The problem contains a part that is analogous to the Σ-problem in the density equation and has not been discussed thus far. Also to be studied is the effect of the strongly diabatic mixed layer on the eddy fields.

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Appendix A. Computation of the geostrophic Reynolds stress

In accordance with expression for the geostrophic component of the eddy velocity field (I.5f), we can present the spectrum of the geostrophic Reynolds stress as follows

$$\mathbf{R}_g(q) = q \int \tilde{\mathbf{R}}_g(\mathbf{q})\,\mathrm{d}\mathbf{n}, \tag{A.1}$$

$$\tilde{\mathbf{R}}_g(\mathbf{q})\delta(\mathbf{q} - \mathbf{q}') = (\mathbf{e}_\rho \times \mathbf{n})(\mathbf{e}_\rho \times \mathbf{n}')\overline{s(\mathbf{q})s^*(\mathbf{q}')}, \tag{A.2}$$

where $\mathbf{n}, \mathbf{n}'$ are the unit vectors in the directions of $\mathbf{q}, \mathbf{q}'$. In the zeroth approximation in the small parameters $\Omega_{1,2}\tau$, eigenvalue equation (I.15,16a) has no external directions and therefore has axi-symmetric solutions. Thus, spectrum (A.2) may be axisymmetric, i.e. proportional to the $2D$ Kroneker tensor within an isopycnal surface $\boldsymbol{\delta}_\rho$ which has no a traceless component (3f). Thus $[\mathbf{R}_0] = 0$. To get non-axisymmetric terms of (A.2) which can yield a non-zero contribution to $[\mathbf{R}]$, we need to expand the field $s(\mathbf{q})$ in (A.2) in powers of $\Omega_{1,2}\tau$

$$s(\mathbf{q}) = s_0(\mathbf{q}) + s_1(\mathbf{q}) + s_2(\mathbf{q}) + \cdots \tag{A.3}$$

As we show in Appendix B, compared to $s_0(\mathbf{q})$, the term $s_1(\mathbf{q})$ contains the phase factor i. Therefore the first order correction $\mathbf{R}_1$ vanishes and so the lowest correction to $\mathbf{R}_0$ is quadratic in $s_1(\mathbf{q})$, i.e. is $\mathbf{R}_2$. At the same time, the term $s_2(\mathbf{q})$ has the same phase as $s_0(\mathbf{q})$ and so the product $s_0 s_2^*$ also yields a contribution to $\mathbf{R}_2$. The detailed computation of the fields $s_1(\mathbf{q}, z)$ and $s_2(\mathbf{q}, z)$ is presented in Appendix B where it is shown that the main contributions into the both fields are barotropic, i.e. corresponding to $n = 0$ in the decompositions like (I.25h). They equal

$$(K_s^0)^{1/2} s_1(\mathbf{q},\rho)/s_0(\mathbf{q},\rho_t) = \mathrm{i}\mathbf{e}_\rho \times \mathbf{I}_1 \cdot \mathbf{n}, \tag{A.4}$$

$$K_s^0 s_2(\mathbf{q},\rho)/s_0(\mathbf{q},\rho_s) = (\mathbf{e}_\rho \times \mathbf{I_1}\cdot\mathbf{n})(\mathbf{e}_\rho \times \mathbf{I_2}\cdot\mathbf{n}) + \mathbf{n}\cdot\mathbf{T_1}\cdot\mathbf{n} + \tau_s\langle \mathrm{sgn} B_1\rangle_0 \mathrm{Im}\{\omega_2\}, \tag{A.5}$$

where the vectors $\mathbf{I_1}, \mathbf{I_2}$ and the tensor $\mathbf{T_1}$ are given in (4b,c), $B_1(\rho)$ is the eigen function of (I.25d) normalized $B_1(\rho_s) = 1$, τ_s is the surface value of the variable τ, (I.16c), ω_2 is the second order correction to ω_1, (1d), which is computed in Appendix B, (B.2), K_s^0 is the surface kinetic energy in the zeroth approximation. Substituting (A.3)–(A.5), (B.2) into (A.2), we get

$$\begin{aligned} K_s^0\mathbf{R}_2(q,z) = \Big(&\frac{1}{4}(1+c_1)\mathbf{I_1}\cdot\mathbf{I_1} + \frac{1}{2}\mathbf{I_1}\cdot\mathbf{I_2} + \frac{3}{2}\mathrm{Trace}(\mathbf{T_1}+\mathbf{T_2})\delta_\rho + \frac{1}{2}(1+c_1)\mathbf{I_1}\mathbf{I_1} \\ &+ \frac{1}{2}(\mathbf{I_1}\mathbf{I_2} + \mathbf{I_2}\mathbf{I_1}) - \mathbf{T_1} - \mathbf{T_2}\Big)E_0(q,\rho_s), \end{aligned} \tag{A.6}$$

where $E_0(q,\rho_s)$ is the eddy energy spectrum on the surface in the zeroth approximation, $\mathbf{T_2}$ is given in (4d). Deriving result (A.6), we have used the following relation:

$$(2\pi)^{-1}\int \mathbf{e}_\rho \times \mathbf{n}\mathbf{e}_\rho \times \mathbf{n}\mathbf{A}\cdot\mathbf{n}\mathbf{B}\cdot\mathbf{n}\,\mathrm{d}\mathbf{n} = \frac{3}{8}\mathbf{A}\cdot\mathbf{B}\boldsymbol{\delta}_\rho - \frac{1}{8}(\mathbf{AB}+\mathbf{BA}), \tag{A.7}$$

where $\mathbf{A}, \mathbf{B}$ are constant vectors. From (A.6) we obtain (4a–d) and (18b).

Appendix B. Computing $s_{1,2}(\mathbf{q})$ in expansion (A.3)

Using (I.14c), one can find expansion (A.3) by computing the expansion $B' = b_0 + b_1 + b_2 + \cdots$ in powers of $\Omega_{1,2}\tau$. In the spirit of the second relation of (I.25h), we will search both $b_{1,2}$ as expansions in the eigen functions $B_n(\rho)$ of the eigen value problem (I.25d). In section 3.2 of paper I we have shown that the decomposition of b_1 is dominated by the barotropic component, i.e. with $n=0$. The result (I.27b,c) is equivalent to (A.4) above.

The equation for the second order perturbation b_2 can be obtained from (I.15,16) straightforwardly, analogously to (I.25c):

$$\hat{L}_0 b_2 = -\mathrm{i}\Omega\tau b_1 + \Omega_1^2\tau^2 b_0 - i\omega_2\tau b_0, \tag{B.1}$$

where ω_2 is the second order correction to the first order result (1d,e); Ω and Ω_1 are given in (I.16d), (18b). To compute ω_2, we multiply (B.1) by $\bar{h}B_1(\rho)$ and integrate over ρ. The integral of the left hand side of (B.1) yields zero since, as pointed out in paper I, the expansions of the functions $b_1, b_2\cdots$ in eigen functions $B_n(\rho)$ do not contain terms with B_1. Then from the integral of the right hand side we deduce with account for (A.4):

$$-\mathrm{i}\omega_2(\mathbf{q})\langle\Gamma(\rho)^{1/2}\rangle_0 = \frac{1}{4}\tau_s(1+\sigma_t)^{-2}\big[(\mathbf{q}\cdot\mathbf{I}_1)^2 - 4\sigma_t^2(\langle\bar{\mathbf{u}} - \mathbf{u}_d\rangle_0\cdot\mathbf{q})^2\big], \tag{B.2}$$

where τ_s is the value of τ at the surface. We use notations (4b,f) as well as the zeroth order of the kinetic energy profile, (18a). We solve equation (B.1) for b_2 as we solved (I.25c). Namely, writing b_2 as a series analogous to (I.25h), and using methodology similar to the one presented in section 3.2 of I, we conclude that the main contribution to b_2 comes from the term $a_0B_0 = a_0$ which can be found by multiplying (B.1) by $\overline{h}$ and then integrating over ρ. Using (1d,e), (I.14c), (18b), (B.2) we arrive at (A.5).

Appendix C. Locality of the eddy energy production

We begin with deriving relation (9c). In accordance with the methodology of the model, we start from computing the density of the flux $\mathbf{F}_B$ in $\mathbf{q}$-space

$$\delta(\mathbf{q}-\mathbf{q}')\tilde{\mathbf{F}}_B(\mathbf{q}) = \mathbf{n}\tilde{s}(\mathbf{q})B'^*(\mathbf{q}'), \qquad \mathbf{n} = \mathbf{q}/q, \tag{C.1}$$

where we account for that in $\mathbf{q}$-space the a-geostrophic component of eddy velocity equals $\tilde{\mathbf{s}} = \mathbf{n}\tilde{s}$. Substituting here relations (I.14c) for B' and (I.14a) for $\tilde{s}$ (with account for the second relation (1d)), we get

$$\delta(\mathbf{q}-\mathbf{q}')\tilde{\mathbf{F}}_b(\mathbf{q}) = \mathbf{n}\mathbf{n}\cdot(\bar{\mathbf{u}}-\mathbf{u}_d+\mathbf{c})\rho s(\mathbf{q})s^*(\mathbf{q}'). \tag{C.2}$$

Integrating the result over $\mathbf{q}'$ and $\mathbf{n}$, we take into account that in the zeroth approximation in $\Omega_{1,2}\tau$ the field $s(\mathbf{q})$ is isotropic, i.e. $s(\mathbf{q}) = s(q)$, and yields the main contribution into energy spectrum $E(q)$. Using also relation (6b) we get the spectrum of flux (9c)

$$\mathbf{F}_B(q) = \rho(\bar{\mathbf{u}}-\mathbf{u}_d+\mathbf{c})E(q). \tag{C.3}$$

Integrating this relation over q, we come to (9c). Substituting the latter into (9b) and further into (9a), we conclude that the contribution of $\mathbf{F}_B$ into (9a) is of the same order as the advection term (the second term in the left hand side of (9a))which, in tern, is of the same order as the advection term in the potential energy equation which is obtained by multiplying (7d) by N^2. Let us evaluate, for example, the contribution of the advection term of (9a) to the corresponding column integral. Assuming $\bar{u} \sim 10^{-2}\,\mathrm{ms}^{-1}$, $\nabla_\rho \sim 1/L \sim 10^{-6}\,\mathrm{m}^{-1}$, $\frac{1}{2}u'^2 = K \sim 10^{-3}\mathrm{m}^2\mathrm{s}^{-2}$ and the dynamical ocean depth $\sim 1\,\mathrm{km}$, we deduce the column contribution $\sim 10^{-8}\mathrm{m}^3\mathrm{s}^{-3}$ that should be compared with the column eddy production (11a) which is valuated below (13e) to be $\sim 10^{-6}\mathrm{m}^3\mathrm{s}^{-3}$. Thus, we conclude that the advection terms are small in comparison with the eddy production which may be considered as local.

Appendix D. Deriving eddy potential energy in terms of large scale fields

In the lowest order of $\Omega\tau$ the eddy field B' is proportional to the eigenfunction B_1 of (I.25b), i.e.

$$B' = AB_1. \tag{D.1}$$

To compute A under the condition that the normalization is $B_1(\rho_s) = 1$, we express eddy kinetic energy in terms of B' with use of (I.5e), (7e), (14c) and (1d)

$$K = \frac{1}{2}\sigma_t^{-1}(r_d\rho_0 f)^{-2}\overline{B'^2} \tag{D.2}$$

and substitute here (D.1). We get

$$K(\rho) = \frac{1}{2}\sigma_t^{-1}(r_d\rho_0 f)^{-2}A^2B_1^2(\rho) = K_sB_1^2(\rho) \tag{D.3}$$

that yields

$$A^2 = 2\sigma_t(r_d\rho_0 f)^2 K_s. \tag{D.4}$$

Substituting (D1,4) into (7a) with account for (I.6d), we obtain (13e).

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